Central Extensions of Linear Algebraic Groups, K-Theory and Homotopy Theory*

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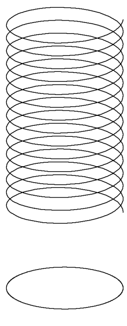
In classical covering space theory we have an isomorphism of the fundamental group with the fibre of the universal cover over the basepoint. Covering spaces of topological groups are group extensions, but not every group extension is a covering space. Perfect groups admit a universal central extension and the kernel of this extension is also called fundamental group. For simply connected Chevalley-groups over a perfect field, this fundamental group, classically called second unstable K-Theory, is exactly the fundamental group of a simplicial resolution. The loops are described explicitly by matrices.

1 Covering Spaces

Example 1.1. *The universal covering space of the circle S^1 is $\mathbb{R}$, with covering map*



$$\begin{tikzcd} \mathbb{R} \arrow[d, two heads] & \ni & x \arrow[d, mapsto] \\ S^1 & \ni & e^{2\pi i x} \end{tikzcd}$$

*This is an extended abstract, without proofs, of the author's german diploma thesis, supervised by Matthias Wendt at Albert-Ludwigs-Universität Freiburg, to whom the author is deeply indebted.

This is a continuous group homomorphism of topological groups, where we think of S^1 as unit complex numbers and of $\mathbb{R}$ as an additive group. The kernel of this map is $\mathbb{Z}$, a discrete abelian group. If we take the neutral element $1 \in S^1$ as basepoint, the fibre over the basepoint $\pi^{-1}(1)$ is exactly the kernel of the group homomorphism π.

The fundamental group $\pi_1(S^1, 1)$ is related to $\pi^{-1}(1)$: We can lift a loop in S^1 to a path in $\mathbb{R}$ which starts at 0 and ends at some point in the fibre. To reverse this process, we can choose an arbitrary path connecting 0 and a point in the fibre, since $\mathbb{R}$ is simply connected, like every universal cover. Homotopy lifting theory shows that this gives an isomorphism

$$\pi_1(S^1, 1) \xrightarrow{\sim} \operatorname{Ker}(\pi).$$

We can also describe the map π as a central group extension

$$\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow S^1$$

which is nothing but a short exact sequence of groups, such that the kernel is mapped into the centre of the middle-term group.

What we have seen here can be vastly generalized:

Theorem 1.2. *For a path-connected topological group G, there exists a universal cover E, unique up to unique isomorphism. Every universal cover admits a unique topological group structure, such that the covering map is a group homomorphism. Choosing identity elements as base-points, the fibre over the basepoint is the kernel and is canonically isomorphic to the fundamental group of G. $\pi_1(G, 1)$ is a discrete, central subgroup of E.*

The isomorphism can bee seen as part of the fibre sequence

$$\cdots \to \Omega \operatorname{Ker}(\pi) \to \Omega E \to \Omega G \to \operatorname{Ker}(\pi) \to E \twoheadrightarrow G$$

which yields a long exact sequence under π_0

$$0 = \pi_1(E) \to \pi_1(G) \to \operatorname{Ker}(\pi) = \pi_0(\operatorname{Ker}(\pi)) \to \pi_0(E) = 0.$$

Example 1.3. *The universal covering space of the Lie group $\mathrm{SL}_2(\mathbb{R})$, commonly just denoted $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, carries a unique group structure such that the covering map $\pi : \widetilde{\mathrm{SL}}_2(\mathbb{R}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{R})$ becomes a group homomorphism. The fundamental group of $\mathrm{SL}_2(\mathbb{R})$ is again $\mathbb{Z}$ and we get a group extension*

$$\mathbb{Z} \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{R}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{R}).$$

The universal cover admits no non-trivial covering spaces by definition, so it is natural to ask whether it admits non-trivial group extensions.

This question has been asked at MathOverflow recently [Hn11].

Example 1.4. *The Lie group $\mathrm{SL}_2(\mathbb{C})$ is simply connected, so it is its own universal cover.*

Example 1.5. *We can look at* SL_2 *as an affine algebraic group. In this setting, the étale fundamental group plays the role previously played by the fundamental group in the euclidean topology. We have* $\pi_1^{\acute{e}t}(\mathrm{SL}_2) = 0$*, so the group is already simply connected. This corresponds to the fact that there are no non-trivial étale isogenies, which is the corresponding notion for covering spaces. In contrast, the affine algebaic group* PSL_2 *is covered by* SL_2*, thus has non-trivial étale fundamental group.*

Remark 1.6. When studying central group extensions of a topological or algebraic group, one can first study all covering spaces up to the universal cover, which is done by the classical or the étale funamental group. Then one can study all central extensions of the universal cover which together yields all central extensions of the original group. Since classical fundamental groups and étale fundamental groups are well understood to some extent, we will now look at simply connected groups.

2 Central Group Extensions

Definition 2.1. A group G is called *perfect*, if it is equal to its own derived group $D(G) = [G, G]$.

Being perfect is somehow the opposite of being abelian.

Definition 2.2. A central extension $E \twoheadrightarrow G$ is *universal*, if every other central extension admits a unique extension homomorphism from E.

Theorem 2.3. *If a group* G *admits a universal central extension* $E \twoheadrightarrow G$*, then both* G *and* E *have to be perfect.*

This can be easily seen by building extensions by $G/D(G)$ or $E/D(E)$ which admit more than one homomorphism from E.

Remark 2.4. For a group extension

$$A \hookrightarrow E \twoheadrightarrow G$$

there is a canonical G-operation on A, by conjugation in E, using a set-theoretic section of $E \twoheadrightarrow G$. This operation is trivial if and only if A sits in the centre of E.

Theorem 2.5. *Group extensions of* G *with* A *with prescribed* G*-Operation on* A *are classified by second group cohomology* $H^2(G, A)$*. The first integral group homology* $H_1(G, \mathbb{Z})$ *is isomorphic to* $G/D(G)$*, so if* G *is perfect,* $H_1(G, \mathbb{Z})$ *vanishes and the universal coefficient theorem*

$$\mathrm{Ext}(H_1(G, \mathbb{Z}), A) \hookrightarrow H^2(G, A) \twoheadrightarrow \mathrm{Hom}(H_2(G, \mathbb{Z}), A)$$

yields an isomorphism. The kernel of the universal central extension is $H_2(G, \mathbb{Z})$*.*

Theorem 2.6 (Hopf, '42)**.** *If a group* G *is perfect, there exists a universal central extension* $E \twoheadrightarrow G$*. Presenting* $G = F/R$ *with a free group* F *and relations* R*, it is given by the formula*

$$E := [F, F]/[F, R]$$

with the map to G being reduction modulo R. The kernel of this map is then

$$H_2(G,\mathbb{Z}) = (R \cap [F,F])/[F,R].$$

3 Chevalley groups

Definition 3.1. A smooth connected linear algebraic group $\mathbb{G}$ over a field k is called *Chevalley-group* if $\mathbb{G}$ is semi-simple, almost simple and split over k. The group $\mathbb{G}$ is called *simply connected* if $\pi_1^{\acute{e}t}(\mathbb{G}) = 0$.

A good example of a simply connected Chevalley-group is SL_n for some n.

Definition 3.2. $\mathbb{G}$ simple means simple Lie algebra and $\mathbb{G}$ almost simple means, there is at most a finite center $Z \subset \mathbb{G}$ and the quotient by the center is simple. $\mathbb{G}$ semi-simple means semi-simple Liea algebra. $\mathbb{G}$ split means any maximal torus is k-isomorphic to a product of copies of $\mathbb{G}_\mathrm{m}$.

Proposition 3.3. *The k-rational points of semi-simple linear algebraic groups over a field are perfect, i.e. they equal their own derived subgroup:*

$$[\mathbb{G}(k), \mathbb{G}(k)] = \mathbb{G}(k).$$

While the derived group as an algebraic subgroup is perfect, the R-rational points of $\mathbb{G}$ are not perfect for every ring R.

Proposition 3.4. *For every (irreducible, reduced) root system Φ, there exists a simply connected Chevalley-group, unique up to unique isomorphism, which we will denote by $\mathbb{G}(\Phi)$.*

Theorem 3.5 (Steinberg, '62)**.** *Let $\mathbb{G}$ be a simply connected Chevalley-group over k infinite with a chosen maximal torus. Then for all roots $\alpha \in \Phi$, there are morphisms $x_\alpha : \mathbb{G}_\mathrm{a} \to \mathbb{G}$ such that $\mathbb{G}(k)$ is presented by generators*

$$\{x_\alpha(u) \mid \alpha \in \Phi,\ u \in k\},$$

$$\big(\text{for } u \in k^\times : \qquad w_\alpha(u) := x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u), \qquad h_\alpha(u) := w_\alpha(u)w_\alpha(1)^{-1}\big)$$

and relations

$$\text{(R1)} \qquad \forall u, v \in k: \quad x_\alpha(u)x_\alpha(v) = x_\alpha(u+v)$$

$$\text{(R2 i)} \qquad \forall \beta \in \Phi : \alpha + \beta \neq 0 \ \forall u, v \in k: \quad [x_\alpha(u), x_\beta(v)] = \prod_{i,j \in \mathbb{N}} x_{i\alpha + j\beta}(N_{\alpha,\beta:i,j} u^i v^j)$$

$$\text{(R2 ii)} \qquad \forall u \in k^\times \ \forall v \in k: \quad w_\alpha(u)x_\alpha(v)w_\alpha(u)^{-1} = x_{-\alpha}(-u^{-2}v)$$

$$\text{(R3)} \qquad \forall u, v \in k^\times: \quad h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$$

where the "structure constants" $N_{\alpha,\beta:i,j} \in \mathbb{Q}$ can be determined, but this is not important for now. The elements $w_\alpha(u)$ represent elements of the Weyl group of $\mathbb{G}$ and the elements $h_\alpha(u)$ are points of the chosen torus.

Example 3.6. *With the notation from Steinberg's presentation, we have in* $\mathrm{SL}_2(k)$ *with the diagonal matrices as maximal torus and* $\alpha := (1,2)$*:*

$$x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad x_{-\alpha}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

$$w_\alpha(t) = \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix}, \qquad h_\alpha(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

Remark 3.7. Steinberg's presentation, together with Hopf's formula suggest a presentation for the universal central extension of a simply connected Chevalley-group, which was also given by Steinberg: Just take the Steinberg-presentation of $\mathbb{G}$ and leave out the relation (R3) for commuting torus elements. The resulting group, as proved by Steinberg, is indeed the universal central extension and we call this the *Steinberg-group* of $\mathbb{G}$, or simply of the root system Φ and the field k. Observe that the Steinberg group is a functor on rings by putting ring elements in the generators x_α. Also note that the elements $x_\alpha(u)$ don't generate $\mathbb{G}(R)$ for any ring R, instead they form a subgroup.

Definition 3.8. Denote the Steinberg-group of $\mathbb{G}(\Phi)$ by $\mathrm{St}(\Phi)$ and its generators by $\widetilde{\mathrm{x}}_\alpha$ to distinguish them from the generators x_α of $\mathbb{G}$. We will write $\mathrm{St}_n := \mathrm{St}(A_{n-1})$ for the Steinberg-group corresponding to $\mathbb{G}(A_{n-1}) = \mathrm{SL}_n$. The subgroup of $\mathbb{G}(\Phi, R)$ generated by the $x_\alpha(u)$ with $u \in R$ is called *elementary subgroup* $E(\Phi, R)$.

4 K-Theory

Definition 4.1. Algebraic K-Theory of a ring R is defined by taking the Grothendieck-group of projective modules over R. First algebraic K-Theory is defined by taking the Whitehead-group $\mathrm{GL}(R)/E(R)$, where $E(R)$ denotes the group generated by elementary matrices. Second algebraic K-Theory is defined by taking the kernel of the extension $\mathrm{St}_\infty(R) \to \mathrm{SL}_\infty(R)$, where the map is the projective limit over all $\mathrm{St}_n(R) \twoheadrightarrow \mathrm{SL}_n(R)$.

Definition 4.2. Unstable first and second K-Theory of a ring R and a root system Φ are defined as

$$\mathrm{K}_1(\Phi, R) := \mathbb{G}(\Phi, R)/E(\Phi, R), \qquad \mathrm{K}_2(\Phi, R) := \mathrm{Ker}\left(\mathrm{St}(\Phi, R) \twoheadrightarrow \mathbb{G}(\Phi, R)\right).$$

Remark 4.3. For an infinite field k, Steinberg's presentation says that $\mathbb{G}(\Phi, k)$ is generated by $E(\Phi, k)$, so $\mathrm{K}_1(\Phi, k) = 0$.

Theorem 4.4 (Matsumoto, '69)**.** *The second unstable* K*-Theory of an infinite field can be presented by symbols* $[\![a, b]\!]$ *with* $a, b \in k^\times$ *under the relations*

(1) $[\![a, bc]\!] + [\![b, c]\!] = [\![ab, c]\!] + [\![a, b]\!]$, *(weak bilinearity)*

(2) $[\![1, 1]\!] = 0$, $[\![a, b]\!] = [\![a^{-1}, b^{-1}]\!]$ *(weak normalization)*

(3) $[\![a, b]\!] = [\![a, (1-a)b]\!]$ *for* $a \neq 1$ *(Steinberg-relation)*

and for non-symplectic root systems ($\Phi \neq C_n$ for all n), the additional relation

$$\llbracket ab, cd \rrbracket = \llbracket a, c \rrbracket + \llbracket a, d \rrbracket + \llbracket b, c \rrbracket + \llbracket b, d \rrbracket. \text{ (strong bilinearity)}$$

Definition 4.5. The group resulting from the relations for non-symplectic root systems is now called $\mathrm{K}_2^M(k)$, the second Milnor-K-Theory of a field. We might call the group resulting for symplectic root systems *symplectic* K*-Theory* and just note that it is isomorphic to the so-called *Milnor-Witt*-K*-Theory* of Morel.

Example 4.6. *For a finite field k, Milnor* K*-Theory* $\mathrm{K}_2^M(k)$ *vanishes.*

5 Homotopy Theory

We will now take a simply connected Chevalley-group, put it in some abstract machine to produce a simplicial group (which you may think of as a combinatorial model of a topological group), do the same for the corresponding Steinberg-group and get a covering of simplicial groups. In this setting, under a reasonable assumption on K-Theory regularity, lifting arguments from the classical covering space theory can be copied to get an isomorphism between the fibre over the basepoint (which is the unstable second K-Theory) and the simplicial fundamental group, represented explicitly by matrices whose entries are polynomials.

Definition 5.1. One can define a simplicial ring $\mathbb{Z}[\Delta^\bullet]$ with graded pieces $\mathbb{Z}[\Delta^n] = \mathbb{Z}[t_1, \ldots, t_n]$ and "reasonable" boundary and face maps

$$d_i : \mathbb{Z}[t_0, \ldots, t_n] \Big/ \left(\sum_{j=0}^{n} t_j - 1 \right) \to \mathbb{Z}[t_0, \ldots, t_{n-1}] / \left(\sum_{j=0}^{n-1} t_j - 1 \right),\ t_j \mapsto \begin{cases} t_j & j < i \\ 0 & j = i \\ t_{j-1} & j > i \end{cases}$$

$$s_j : \mathbb{Z}[t_0, \ldots, t_n] \Big/ \left(\sum_{i=0}^{n} t_i - 1 \right) \to \mathbb{Z}[t_0, \ldots, t_{n+1}] / \left(\sum_{i=0}^{n+1} t_i - 1 \right),\ t_i \mapsto \begin{cases} t_i & i < j \\ t_i + t_{i+1} & i = j \\ t_{i+1} & i > j \end{cases}$$

For a field k, this yields a simplicial k-algebra $k[\Delta^\bullet] := \mathbb{Z}[\Delta^\bullet] \otimes k$. For any functor $\mathcal{F}$ on k-algebras, with values in some category $\mathcal{C}$, we can take values in this simplicial k-algebra and get a simplicial object $\mathcal{F}(k[\Delta^\bullet])$ in $\mathcal{C}$. Now we define the *singular resolution* of $\mathcal{F}$

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1} \mathcal{F} := \left(A \mapsto \mathcal{F}(k[\Delta^\bullet] \underset{k}{\otimes} A) \right).$$

To talk about simplicial fundamental groups and homotopy theory, one does not only need simplicial objects, but also a model category structure. We take the usual Kan model structure, which is essentially the model structure which yields the Serre model structure on CW-complexes after geometric realization. If you don't know anything about model structures, take the next theorem as a definition and everything will be fine.

Theorem 5.2. *Simplicial groups are fibrant in the Kan model category, so we can calculate the fundamental group of a simplicial group $G_\bullet$ by loops modulo based homotopies*

$$\pi_1(G_\bullet, \mathrm{id}) = \frac{\{\alpha \in G_1 \mid d_0\alpha = d_1\alpha = \mathrm{id}\}}{\{d_0\sigma \sim d_1\sigma \mid \sigma \in G_2 : d_2\sigma = s_0\,\mathrm{id}\}}.$$

Corollary 5.3. *For a simply connected Chevalley-group $\mathbb{G}$, we can calculate the fundamental group of its singular resolution*

$$\pi_1(\mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{G})(k), \mathrm{Id}) = \frac{\{M \in \mathbb{G}(k[t]) \mid M(0) = M(1) = \mathrm{Id}\}}{\{N(1-t,t) \sim N(0,t) \mid N \in \mathbb{G}(k[t,s]) : N(t,0) = \mathrm{Id}\}}.$$

Theorem 5.4 (Jardine, '83)**.** *The fundamental group of the singular resolution of a simply connected Chevalley-group $\mathbb{G}$ is isomorphic to an unstable second Karoubi-Villamayor-*K*-group.*

These K-groups are isomorphic to the unstable K-groups defined above (at least for fields) and Jardine's proof proceeds by concatenating isomorphisms where little is known about the actual structure of the resulting isomorphism map.

Theorem 5.5. *If second unstable* K*-Theory has regularity in low degrees, i.e.*

$$\mathrm{K}_2(\Phi, k[t_1, t_2]) \simeq \mathrm{K}_2(\Phi, k[t_1]) \simeq \mathrm{K}_2(\Phi, k),$$

then the fundamental group of the singular resolution of a simply connected Chevalley-group $\mathbb{G}$ is isomorphic to an unstable second K*-group by explicitly mapping Steinberg-symbols $\{a, b\}$ in* K*-Theory to loops $C_t^\alpha(a, b) \in \mathbb{G}(k[t])$ which are defined by*

$$X_t^\alpha(u) := x_\alpha(tu),$$

$$W_t^\alpha(u) := X^\alpha(u)X^{-\alpha}(-u^{-1})X^\alpha(u),$$
$$H_t^\alpha(u) := W_t^\alpha(u)W_t^\alpha(1)^{-1},$$
$$C_t^\alpha(a, b) := H_t^\alpha(a)H_t^\alpha(b)H_t^\alpha(ab)^{-1}.$$

To prove that this map $\mathrm{K}_2^M(k) \to \pi_1(\mathrm{Sing}_\bullet^{\mathbb{A}^1}\,\mathbb{G}(k))$ is an isomorphism, one needs two important ingredients:

1. A factorization lemma for Chevalley-groups over polynomial rings, which allows to decompose every element into a product of elementary matrices - essentially due to Suslin and Vorst for SL_n, extended to general Chevalley-groups by Abe and Wendt.

2. A homotopy lifting lemma for the singular resolution of the Steinberg-group, which depends on regularity for second unstable K-Theory. This, sadly, is not very well available in the literature yet, so it stays somewhat conjectural.

The rest is done by proving an analogue of Chevalley's commutator formula in the simplicial group, which then allows to copy large parts of Matsumoto's proof and classical covering space theory. $\lozenge$

Example 5.6. *If we take $\alpha = (1,2)$ in $\Phi = A_1$, then $\mathbb{G}(\Phi) = \mathrm{SL}_2$ and*

$$X_t^{\alpha}(u) = x_{\alpha}(tu) = \begin{pmatrix} 1 & tu \\ 0 & 1 \end{pmatrix}, \qquad X_t^{-\alpha}(u) = x_{-\alpha}(tu) = \begin{pmatrix} 1 & 0 \\ tu & 1 \end{pmatrix}$$

paths from Id *to $x_{\alpha}(u)$ resp. $x_{-\alpha}(u)$. Now $W_t^{\alpha}(u) \neq w_{\alpha}(tu)$, instead*

$$W_t^{\alpha}(u) = \begin{pmatrix} 1 & tu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -tu^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & tu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-t^2 & (2-t^2)tu \\ -tu^{-1} & 1-t^2 \end{pmatrix}$$

so we have a path from Id *to $w_{\alpha}(u)$. Similarly,*

$$H_t^{\alpha}(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(1-u)t}{u} \begin{pmatrix} (t^2-2)tu & -(t^2-2)(t^2-1)u \\ t^2-1 & -(t^2-2)t \end{pmatrix}$$

Example 5.7. *Finally...*

$$\begin{aligned} \mathrm{C}_{\mathrm{t}}^{\alpha}(x,y) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ t(t^2-1)(t^2-2)\frac{(1-x)(1-y)}{x^2y} \begin{pmatrix} xt(t^2-1)(1-x) & -yx^2\left((t^2-1)^2(1-x)+x\right) \\ \frac{(t^2-1)^2(1-x)-1}{t^2-2} & -xyt(t^2-1)(1-x) \end{pmatrix}, \end{aligned}$$

6 $\mathbb{A}^1$-Homotopy Theory

This section is only of interest, if you know what Morel&Voevodsky's $\mathbb{A}^1$-Homotopy Theory of Schemes is about.

Theorem 6.1 (Wendt, '10)**.** *Singular resolutions of Chevalley groups have the affine Brown-Gersten-property for the Nisnevich topology, which allows to compute their fundamental group presheaf over affine schemes just by taking the fundamental group of the sections, so*

$$\pi_1^{\mathbb{A}^1}(\mathbb{G})(k) = \pi_1((\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}\mathbb{G})(k)).$$

This tells us, that the fundamental group computed above is the $\mathbb{A}^1$-fundamental group of $\mathbb{G}$ over k. For $\mathbb{G}$ of rank ≥ 3, this is another proof of a theorem already known:

Theorem 6.2 (Wendt, '10)**.** *The $\mathbb{A}^1$-fundamental group of a simply connected Chevalley-group of rank ≥ 3 over an infinite field k is isomorphic to $\mathrm{K}_2^{MW}(k)$ for symplectic groups and $\mathrm{K}_2^{M}(k)$ otherwise.*

Remember (from above) that $\mathrm{K}_2^{MW}(k)$ is isomorphic to the symplectic K-Theory which arises as presentation for symplectic root systems in Matsumoto's theorem. We note that the isomorphism from $\mathrm{K}_2^{MW}(k)$ to the symplectic K-Theory requires the Milnor conjecture (i.e. Voevodsky's norm residue isomorphism theorem) as far as we know.

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